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1993 J. Phys. A: Math. Gen. 26 433

(http://iopscience.iop.org/0305-4470/26/2/026)

J. Phys. A: Math. Gen. 26 (1993) 433–442. Printed in the UK

Global d-invariance in field theory

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Received 14 February 1992

Abstract. In a previous work we derived a remarkable *local* identity which allows writing of any Lagrangian as a 'linear combination' of its field equations plus a divergence. Using this identity we were able to provide an alternative proof of the fact that a (higher-order) Lagrangian has identically vanishing field equations if and only if it is *locally* a divergence. The aim of this work is to investigate how far we can go in globalizing the previous results for (higher-order) Lagrangians. In the case of vector or affine bundles the previous results admit global generalizations in a natural way. The true obstacle is the topological structure of the fibre bundle (both of the basis manifold and of the fibres). As a general rule, it turns out that we can globalize in a non-unique way the previous results when the fibre bundle admits global sections and, moreover, it is contractible by fibred morphisms to one of its global sections. Uniqueness is lost at the level of affine bundles, and for 'non-trivial' topologies we lose the globality of the result.

Introduction

It is well known that a Lagrangian density of the form $\mathscr{L} \equiv \mathrm{d}_\mu \mathscr{F}^\mu$, where d_μ is the total (or formal) derivative with respect to x^μ, and $\mathscr{F}^\mu$ is an arbitrary function of the fields and their derivatives up to an arbitrary order, yields identically vanishing field equations $\delta_a \mathscr{L} \equiv 0$. This can be verified just by direct replacement of $\mathscr{L} \equiv \mathrm{d}_\mu \mathscr{F}^\mu$ into $\delta_a \mathscr{L}$. Accordingly, two Lagrangians differing by a divergence yield the same field equations; they are called d-*equivalent*, while the Lagrangian formalism is said to be d-*invariant*. The previous fact involves a sufficient condition for a Lagrangian to have identically vanishing field equations. An important problem is to show that the condition is also *locally* necessary, i.e. to show that $\delta_a \mathscr{L} \equiv 0$, implies $\mathscr{L} \equiv \mathrm{d}_\mu \mathscr{F}^\mu$.

The necessary condition can be easily obtained in classical mechanics, as well as for its higher-order generalizations, and also for first-order field theory [1]. A classical constructive method of proof, which can be found in [1], consists in explicitly writing the total derivatives involved in the field equations. The terms containing derivatives of each different order must be independently zero. This means that the factors multiplying them must be zero, giving differential conditions which might be integrated to obtain the explicit structure of the Lagrangian.

The problem, however, complicates very much for orders larger than one if there are several independent variables (see [2]). An implicit proof that $\delta_a \mathscr{L} \equiv 0$ is equivalent to the Lagrangian being a divergence was given by Krupka [3]. For higher-order

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0305-4470/93/020433+10$07.50

Lagrangians the generalization of the constructive method of proof of [1] becomes practically unmanageable. However, a new constructive method must be looked for, since, in view of applications to theoretical physics, it is still interesting to derive a method for explicitly writing as a divergence a given Lagrangian having identically vanishing field equations.

Inspired by some previous results by Vainberg [4], Atherton and Homsy [5] and Engels [6], in the framework of higher-order field theory we derived [7] a local identity which allows any Lagrangian to be written locally as a 'linear combination' of its field equations plus a divergence. The identity found in [7] was applied to solve two problems: first, to show that the necessary and sufficient condition for identically vanishing field equations is that the Lagrangian is locally a divergence; second, to study the possibility of removing from field theory the ambiguities related to the non-invariance of the energy-momentum tensor under the addition of a divergence to the Lagrangian.

Here we look for the global extension of the identity we found in [7] and we discuss a generalization to the cases of non-trivial fibre topology. The previous extension is important not only from the mathematical viewpoint, but also in view of its applications to theoretical physics.

This paper is organized as follows. Section 1 is essentially a review of previous results and fixes the notation (fibred manifolds and prolongations in section 1.1, calculus of variations in section 1.2 and fibred homotopy formula in section 1.3). In section 2 we consider the globalization of the previous formalism. In section 2.1 we consider the case of vector bundles and in section 2.1.1 we show explicitly how to calculate the relevant objects in a covariant way by means of connections. The case of affine bundles is considered in section 2.2, while in section 2.2.1 we apply the previous results to electromagnetic fields on curved manifolds.

1. Preliminaries

Since the fields describing our physical systems are assumed to be (local) sections of a fibred manifold† $\pi : B \to M$, over an orientable basis manifold‡ M, we shall start by recalling the fundamentals of calculus of variations in fibred manifolds, and by fixing the notation.

1.1. Fibred manifolds and prolongations

As is usual in this kind of problem, we shall work only with *fibred coordinates* (x^μ, y^a) over B, where $\mu = 1, \ldots, m = \dim(M)$ and $a = 1, \ldots, n = \dim(B) - \dim(M)$. Without any loss of generality, we may consider only fibred coordinates (x^μ, y^a) ranging in domains Ω of the type $\Omega = U \times V$, where $U \subseteq \boldsymbol{R}^m$ and $V \subseteq \boldsymbol{R}^n$ are open subsets. With such coordinate systems, a (local) section $\phi : M \to B$ will be represented by a map $\phi : (x^\mu) \mapsto (x^\mu, \phi^a(x))$.

† In [7] we assumed that the fibred manifold B was a vector bundle over M. Here we do not make any restriction on B (we do not even ask B to be a bundle over M).

‡ All the manifolds we use in the following will be assumed to be smooth (i.e. C^∞), connected, locally compact and paracompact. All the mappings will be assumed to be smooth.

For any positive integer k, the kth-order prolongation of the fibred manifold $\pi: B \to M$, which is a fibred manifold $\pi^k: J^k B \to M$, and the kth-order prolongation of each (local) section $\phi: M \to B$, which is a (local) section $j^k\phi: M \to J^k B$, are defined in a canonical way. On $J^k B$ we have canonical fibred coordinates obtained by taking prolongations of fibred coordinates of B, which are the only coordinates we shall use in $J^k B$. The kth-order prolongation of a fibred coordinate system (x^μ, y^a), with domain Ω, will be denoted by $(x^\mu, y^a, y^a_{\mu_1}, \ldots, y^a_{\mu_1 \ldots \mu_k})$, and its domain by Ω_k. Using canonical fibred coordinates in $J^k B$, the kth-order prolongation $j^k\phi: M \to J^k B$ of a (local) section $\phi: M \to B$ will be represented by the map $j^k\phi: (x^\mu) \mapsto (x^\mu, \phi^a(x), \partial_{\mu_1}\phi^a(x), \ldots, \partial_{\mu_1 \ldots \mu_k}\phi^a(x))$, where we use the standard notation

$$\partial_{\mu_1 \ldots \mu_s}\phi^a(x) = \frac{\partial^s \phi^a(x)}{\partial x^{\mu_1} \ldots \partial x^{\mu_s}} \qquad \text{with } 1 \leqslant s \leqslant k$$

for partial derivatives of the functions $\phi^a(x)$.

Remark. In order to avoid unnecessary complications, we shall assume that for $2 \leqslant s \leqslant k$ all the 'coordinates' $y^a_{\mu_1 \ldots \mu_s}$ are symmetric in the indices $(\mu_1 \ldots \mu_s)$, with $1 \leqslant \mu_i \leqslant m$ for $1 \leqslant i \leqslant s$. Such a convention involves the explicit symmetrization of certain indexed objects, but leads to formulae which are simpler than those obtained using true coordinates (e.g. multi-indices or increasing sequences of indices $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_s$).

Let $f: \Omega_k \to \boldsymbol{R}$ be a real-valued function defined on the domain of a canonical fibred chart $(x^\mu, y^a, y^a_{\mu_1}, \ldots, y^a_{\mu_1 \ldots \mu_k})$ of $J^k B$. The formal derivative $\mathrm{d}_\mu f: \Omega_{k+1} \to \boldsymbol{R}$ is then defined by

$$\mathrm{d}_\mu f = \partial_\mu f + y^a_\mu \partial_a f + \sum_{s=1}^{k} y^a_{\nu_1 \ldots \nu_s \mu} \partial_a^{\nu_1 \ldots \nu_s} f \tag{1}$$

where the partial derivatives $\partial_\mu f, \partial_a f, \partial_a^\mu f, \ldots, \partial_a^{\mu_1 \ldots \mu_k} f$ are uniquely defined by

$$\mathrm{d}f = \partial_\mu f \, \mathrm{d}x^\mu + \partial_a f \, \mathrm{d}y^a + \partial_a^\mu f \, \mathrm{d}y^a_\mu + \ldots + \partial_a^{\mu_1 \ldots \mu_k} f \, \mathrm{d}y^a_{\mu_1 \ldots \mu_k}$$

and by the requirement that $\partial_a^{\mu_1 \ldots \mu_s} f$ be symmetric in $(\mu_1 \ldots \mu_s)$ for $2 \leqslant s \leqslant k$.

For higher-order formal derivatives will shall use the same notation we used for higher-order partial derivatives, i.e. $\mathrm{d}_{\mu_1 \ldots \mu_s} f \equiv \mathrm{d}_{\mu_1} \ldots \mathrm{d}_{\mu_s} f$. Recall that $\mathrm{d}_{\mu_1 \ldots \mu_s} f$ is symmetric in the indices $(\mu_1 \ldots \mu_s)$.

1.2. Calculus of variations

We assume that the dynamics of our physical system is described by a kth-order Lagrangian density

$$\mathscr{L} = \mathscr{L}(x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1 \ldots \mu_k}) \tag{2}$$

on a fibred manifold B. The first-variation formula for $\mathscr{L}$ reads then as follows:

$$\delta\mathscr{L} = \delta_a \mathscr{L} \delta y^a + \mathrm{d}_\mu \left[\delta_a^\mu \mathscr{L} \delta y^a + \delta_a^{\mu\nu} \mathscr{L} \delta y^a_\nu + \sum_{s=2}^{k=1} \delta_a^{\mu\nu_1 \ldots \nu_s} \mathscr{L} \delta y^a_{\nu_1 \ldots \nu_s} \right] \tag{3}$$

where the variational derivatives $\delta_a \mathscr{L}, \delta_a^{\mu\nu} \mathscr{L}, \ldots, \delta_a^{\mu\nu_1 \ldots \nu_k} \mathscr{L}$ are recursively defined by

$$\delta_a^{\mu_1 \ldots \mu_k} \mathscr{L} = \partial_a^{\mu_1 \ldots \mu_k} \mathscr{L} \qquad \delta_a^{\mu_1 \ldots \mu_{k-1}} \mathscr{L} = \partial_a^{\mu_1 \ldots \mu_{k-1}} \mathscr{L} - \mathrm{d}_\nu \delta_a^{\mu_1 \ldots \mu_{k-1}\nu} \mathscr{L}$$

$$\ldots \qquad \delta_a^\mu \mathscr{L} = \partial_a^\mu \mathscr{L} - \mathrm{d}_\nu \delta_a^{\mu\nu} \mathscr{L} \qquad \delta_a \mathscr{L} = \partial_a \mathscr{L} - \mathrm{d}_\mu \delta_a^\mu \mathscr{L}. \tag{4}$$

The variational derivatives $\delta_a^{\mu_1 \ldots \mu_s} \mathscr{L}$ are symmetric in $(\mu_1 \ldots \mu_s)$ for $2 \leqslant s \leqslant k$.

The two addenda of the right-hand side of the first variation formula (3) are always globally well defined scalar-densities, independently of the order k of the Lagrangian (2). On the contrary, the vector-density

$$f^\mu(\mathscr{L}) = \delta^\mu_a \mathscr{L} \delta y^a + \delta^{\mu\nu}_a \mathscr{L} \delta y^a_\nu + \sum_{s=2}^{k-1} \delta^{\mu\nu_1\cdots\nu_s}_a \mathscr{L} \delta y^a_{\nu_1\cdots\nu_s}$$

is globally well defined only for orders $k \leqslant 2$. However, as was proved in [8], this problem can be overcome by simply using a linear connection on the basis manifold M. In fact (see [9] and [8] for details), for any linear connection $\Gamma^\alpha_{\beta\mu}$ on M there exists a uniquely defined set of coefficients $\delta^\mu_a(\mathscr{L}, \Gamma)$, $\delta^{\mu\nu}_a(\mathscr{L}, \Gamma), \ldots, \delta^{\mu\nu_1\cdots\nu_{k-1}}_a(\mathscr{L}, \Gamma)$ such that the first variation formula (3) can be re-written as follows:

$$\delta\mathscr{L} = \delta_a \mathscr{L} \delta y^a + \mathrm{d}_\mu \left[\delta^\mu_a(\mathscr{L}, \Gamma) \delta y^a + \delta^{\mu\nu}_a(\mathscr{L}, \Gamma) \delta y^a_\nu + \sum_{s=2}^{k-1} \delta^{\mu\nu_1\cdots\nu_s}_a(\mathscr{L}, \Gamma) \delta y^a_{\nu_1\cdots\nu_s} \right] \tag{5}$$

and the vector-density

$$f^\mu(\mathscr{L}, \Gamma) = \delta^\mu_a(\mathscr{L}, \Gamma) \delta y^a + \delta^{\mu\nu}_a(\mathscr{L}, \Gamma) \delta y^a_\nu + \sum_{s=2}^{k-1} \delta^{\mu\nu_1\cdots\nu_s}_a(\mathscr{L}, \Gamma) \delta y^a_{\nu_1\cdots\nu_s}$$

is globally well defined for any order k (see [9] for the explicit expression at order $k = 3$).

Remarks. The coefficients $\delta^{\mu\nu_1\cdots\nu_s}_a(\mathscr{L}, \Gamma)$ are symmetric in $(\nu_1 \ldots \nu_s)$ for $2 \leqslant s \leqslant k-1$. The highest one

$$\delta^{\mu\nu_1\cdots\nu_{k-1}}_a(\mathscr{L}, \Gamma) \equiv \delta^{\mu\nu_1\cdots\nu_{k-1}}_a(\mathscr{L}) \equiv \partial^{\mu\nu_1\cdots\nu_{k-1}}_a(\mathscr{L})$$

is the only one which is symmetric in all the contravariant indices, is always independent of the connection $\Gamma^\alpha_{\beta\mu}$ for any order k, and is tensorial.

The vector-densities $f^\mu(\mathscr{L}, \Gamma)$ and $f^\mu(\mathscr{L}, \Gamma')$ corresponding to two connections Γ and Γ' differ by the formal divergence $\mathrm{d}_\nu h^{\mu\nu}(\mathscr{L}, \Gamma, \Gamma')$ of a globally well defined skew-symmetric tensor-density $h^{\mu\nu}(\mathscr{L}, \Gamma, \Gamma')$.

At order $k \leqslant 2$ we have $f^\mu(\mathscr{L}, \Gamma) \equiv f^\mu(\mathscr{L})$.

1.3. Fibred homotopy formula

It is well known that field equations $\delta_a \mathscr{L} = 0$ are invariant under the addition of a divergence to the Lagrangian. This property is called d-*invariance*, and in order to preserve d-invariance in field theory one has to select a representative for the family of d-equivalent Lagrangians [7]. Let us recall how the basic local identity for a kth-order Lagrangian $\mathscr{L}$ is obtained.

We assume that the domain Ω of the fibred coordinate system admits a fibred homotopy between the identity and one of its sections ϕ; i.e. there is a map $C : [0, 1] \times \Omega \to \Omega$ of the type $(\tau, x^\mu, y^a) \mapsto (x^\mu, c^a(\tau, x, y))$ and such that

$$C(1, x^\mu, y^a) \equiv (x^\mu, y^a) \qquad \text{and} \qquad C(0, x^\mu, y^a) \equiv (x^\mu, \phi^a(x)).$$

If this is *not* true, we can assume that each fibre of the domain Ω is contractible to a point and consider a smaller domain Ω where the property holds. For any positive integer k, the kth-order prolongation of C is the fibred homotopy $C^k : [0, 1] \times \Omega_k \to \Omega_k$ defined by

$$(\tau, x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1\cdots\mu_k}) \mapsto (x^\mu, c^a(\tau, x, y), \mathrm{d}_\mu c^a(\tau, x, y), \ldots, \mathrm{d}_{\mu_1\cdots\mu_k} c^a(\tau, x, y)).$$

It is clear from its definition that C^k is a homotopy between the identity of Ω_k and the section $j^k\phi$; that is, we have

$$C(1, x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1\ldots\mu_k}) \equiv (x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1\ldots\mu_k})$$

and

$$C(0, x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1\ldots\mu_k}) \equiv (x^\mu, \phi^a(x), \partial_{\mu_1}\phi^a(x), \ldots, \partial_{\mu_1\ldots\mu_k}\phi^a(x)).$$

Given a kth-order Lagrangian $\mathscr{L}$ we consider the one-parameter family $\mathscr{L}_\tau$ of Lagrangians defined (over Ω_k) by setting

$$\mathscr{L}_\tau(x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1\ldots\mu_k}) = \mathscr{L}(x^\mu, c^a(\tau, x, y), \mathrm{d}_\mu c^a(\tau, x, y), \ldots, \mathrm{d}_{\mu_1\ldots\mu_k} c^a(\tau, x, y)).$$

This family is such that

$$\mathscr{L}_1(x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1\ldots\mu_k}) \equiv \mathscr{L}(x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1\ldots\mu_k})$$

and

$$\mathscr{L}_0(x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1\ldots\mu_k}) \equiv \mathscr{L}(x^\mu, \phi^a(x), \partial_{\mu_1}\phi^a(x), \ldots, \partial_{\mu_1\ldots\mu_p}\phi^a(x)).$$

Accordingly, if we now apply the global first-variation formula (5) to the derivative $\mathrm{d}/\mathrm{d}\tau(\mathscr{L}_\tau)$ of the family and then we integrate over $\tau \in [0, 1]$, we obtain the following identity:

$$\mathscr{L} = \mathscr{L}_0 + \mathrm{d}_\mu \tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma) + \tilde{\mathscr{L}} \tag{6}$$

where the vector-density $\tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma)$ and the scalar-density $\tilde{\mathscr{L}}$ are defined by

$$\tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma) = \int_0^1 \left\{ \delta^\mu_a(\mathscr{L}_\tau, \Gamma) \frac{\partial c^a}{\partial \tau} + \delta^{\mu\nu}_a(\mathscr{L}_\tau, \Gamma)\, \mathrm{d}_\nu \frac{\partial c^a}{\partial \tau} + \sum_{s=2}^{k-1} \left[\delta^{\mu\nu_1\ldots\nu_s}_a(\mathscr{L}_\tau, \Gamma)\, \mathrm{d}_{\nu_1\ldots\nu_s} \frac{\partial c^a}{\partial \tau} \right] \right\} \mathrm{d}\tau \tag{7}$$

and

$$\tilde{\mathscr{L}} = \int_0^1 \left\{ \delta_a(\mathscr{L}_\tau) \frac{\partial c^a}{\partial \tau} \right\} \mathrm{d}\tau. \tag{8}$$

Although we did not write it explicitly, the quantities $\mathscr{L}_0$, $\tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma)$ and $\tilde{\mathscr{L}}$ depend on the choice of the homotopy C.

Since $\mathscr{L}_0 \equiv \mathscr{L}_0(x)$ is a scalar density on the manifold M, there exists *locally* a vector-density $f^\mu_0(x)$ such that $\mathscr{L}_0(x) \equiv \mathrm{d}_\mu f^\mu_0(x)$. Therefore, we have the local identity

$$\mathscr{L} = \mathrm{d}_\mu \mathscr{F}^\mu + \tilde{\mathscr{L}} \tag{9}$$

where we set $\mathscr{F}^\mu = f^\mu_0(x) + \tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma)$. This is the kind of identity we derived in [7] by using local fibred homotopies of the type $C : (\tau, x^\mu, y^a) \mapsto (x^\mu, \tau y^a)$. We remark that the vector-density $\mathscr{F}^\mu$ is defined only up to a divergence, since we may replace the vector-density $f^\mu_0(x)$ with a vector-density of the form $f^\mu_0(x) + \partial_\nu h^{\mu\nu}_0(x)$, where $h^{\mu\nu}_0(x)$ is a skew-symmetric tensor density.

Using the identity (9) it is now evident that $\delta_a \mathscr{L} \equiv 0$ locally implies $\mathscr{L} \equiv \mathrm{d}_\mu \mathscr{F}^\mu$, in full agreement with the general existence theory (see [2]). We stress, however, that this method provides a constructive way to find, through equation (7), a vector-density $\mathscr{F}^\mu$ whose divergence is the given Lagrangian.

In order to obtain d-invariant quantities we have to pick up a representative for the d-equivalent Lagrangians. Once again, the identity (6) helps us in this task. In fact, as we did in [7], it allows us to select the following representative for the d-equivalent Lagrangians

$$\tilde{\mathscr{L}} \equiv \mathscr{L} - \mathscr{L}_0(x^\mu) - \mathrm{d}_\mu \tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma) \equiv \int_0^1 \left\{ \delta_a(\mathscr{L}_\tau) \frac{\partial c^a}{\partial \tau} \right\} \mathrm{d}\tau \tag{10}$$

which vanishes identically when applied to a pure divergence.

2. Globalization

The calculations of section 1.3 are of local nature, because they rely on the local fibred homotopy C and the corresponding local section ϕ. We shall here generalize these results under the hypothesis that the fibred manifold $\pi : B \to M$ is a vector bundle over M, an affine bundle over M or a slightly more general fibred manifold.

2.1. Vector bundles

In the case of a vector bundle $\pi : B \to M$ over M, there is a special global section of B, the zero section, and a special global homotopy $C : [0, 1] \times B \to B$, which is induced by the vector space structure of the fibres of B, i.e. defined by $C : (t, v) \mapsto tv$.

When dealing with vector bundles, we restrict our attention to fibred charts (x^μ, y^a) coming from local trivializations of the vector bundle B. Accordingly we have domains Ω of the type $\Omega = U \times \boldsymbol{R}^n$ where $U \subseteq \boldsymbol{R}^m$ is an open subset, and transition functions which are linear in the fibre coordinates (y^a). The local representations of the special global homotopy $C : [0, 1] \times B \to B$ in this kind of fibred charts are of the type $(\tau, x^\mu, y^a) \mapsto (x^\mu, \tau y^a)$, and their kth-order prolongations are defined by

$$(\tau, x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1 \ldots \mu_k}) \mapsto (x^\mu, \tau y^a, \tau y^a_\mu, \ldots, \tau y^a_{\mu_1 \ldots \mu_k}). \tag{11}$$

All the relevant terms appearing in formula (6), i.e.

$$\mathscr{L}_0 \qquad \tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma) \qquad \text{and} \qquad \tilde{\mathscr{L}}$$

are globally well defined, so that there is a preferred d-invariant Lagrangian

$$\tilde{\mathscr{L}} \equiv \mathscr{L} - \mathscr{L}_0(x) - \mathrm{d}_\mu \tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma) \equiv \int_0^1 \{ \delta_a(\mathscr{L}_\tau) y^a \} \, \mathrm{d}\tau. \tag{12}$$

The only possible source of non-globality in the vector-density

$$\mathscr{F}^\mu = f_0^\mu(x) + \tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma)$$

resides in the vector-density $f_0^\mu(x)$. In fact, being $\tilde{\mathscr{F}}^\mu(\mathscr{L}, \Gamma)$ global, the vector-density $\mathscr{F}^\mu$ *defined in this way is global if and only if the vector-density* $f_0^\mu(x)$ *is global.* Accordingly, it is now easy to prove that a Lagrangian defined on the sections of a vector bundle B and having identically vanishing field equations is a global divergence if and only if the value of the Lagrangian on the zero section is an exact form.

The possibility of positively answering this problem depends both on the basis manifold M and on the Lagrangian $\mathscr{L}$. In particular, we have a global $\mathscr{F}^\mu$ independently of the topology of M when the Lagrangian $\mathscr{L}_0$ vanishes, or it is an exact m-form. On the other hand, we have a global $\mathscr{F}^\mu$ independently of the Lagrangian $\mathscr{L}$ when the highest cohomology group $H_m(M)$ vanishes. In any case, when $H_m(M)$ does not vanish there exist always Lagrangians $\mathscr{L}$ for which no global $\mathscr{F}^\mu$ can be found.

2.1.1. Covariant first variation formula. We shall see here, as an example, the procedure for constructing the global vector-density $\mathscr{F}^\mu(\mathscr{L}, \Gamma)$ which enters the global first variation formula (5) (for details and further reference see [9]).

Covariant derivatives of sections of the vector bundle B can be defined by choosing a principal connection $A^a_{b\mu}$ on the frame bundle of B, while for defining higher-order covariant derivatives we need to choose also a linear connection $\Gamma^\alpha_{\beta\mu}$ on the basis manifold M (for simplicity, we shall assume that the torsion of $\Gamma^\alpha_{\beta\mu}$ vanishes). Once we have fixed the connections $A^a_{b\mu}$ and $\Gamma^\alpha_{\beta\mu}$, higher-order covariant derivatives of a section $\phi : M \to B$ are defined by repeated use of $A^a_{b\mu}$ and $\Gamma^\alpha_{\beta\mu}$. For example, we have

$$\nabla_\sigma \phi^a = \mathrm{d}_\sigma \phi^a + A^a_{b\sigma}\phi^b$$

$$\nabla_\sigma \nabla_\mu \phi^a = \mathrm{d}_\sigma \nabla_\mu \phi^a + A^a_{b\sigma}\nabla_\mu \phi^b - \nabla_\alpha \phi^a \Gamma^\alpha_{\mu\sigma}$$

$$\nabla_\sigma \nabla_\mu \nabla_\nu \phi^a = \mathrm{d}_\sigma \nabla_\mu \nabla_\nu \phi^a + A^a_{b\sigma}\nabla_\mu \nabla_\nu \phi^b - \nabla_\alpha \nabla_\nu \phi^a \Gamma^\alpha_{\mu\sigma} - \nabla_\mu \nabla_\alpha \phi^a \Gamma^\alpha_{\nu\sigma}$$

and so on. We recall that for any positive integer k this defines a one-to-one correspondence between partial derivatives $(x^\mu, \phi^a(x^\nu), \phi^a_{\mu_1}(x^\nu), \ldots, \phi^a_{\mu_1\ldots\mu_k}(x^\nu))$ and symmetrized covariant derivatives $(x^\mu, \phi^a(x^\nu), \nabla_{\mu_1}\phi^a(x^\nu), \ldots, \nabla_{\mu_1\ldots\mu_k}\phi^a(x^\nu))$ of sections of the vector bundle B. This one-to-one correspondence depends only on the connections $A^a_{b\mu}$ and $\Gamma^\alpha_{\beta\mu}$ (and not on the particular section ϕ of B):

$$(x^\mu, y^a, y^a_{\mu_1}, \ldots, y^a_{\mu_1\ldots\mu_k}) \leftrightarrow (x^\mu, y^a, \nabla_{\mu_1} y^a, \ldots, \nabla_{\mu_1\ldots\mu_k} y^a). \tag{13}$$

Let us then consider a kth-order Lagrangian (2), and let us assume that $\mathscr{L}$ has been already expressed in terms of symmetrized covariant derivatives of the fields by means of (13):

$$\mathscr{L} = \mathscr{L}(x^\mu, y^a, y^a_\mu, \ldots, y^a_{\mu_1\ldots\mu_k}) \equiv \mathscr{L}'(x^\mu, y^a, \nabla_{\mu_1} y^a, \ldots, \nabla_{\mu_1\ldots\mu_k} y^a). \tag{14}$$

Then, the first variation formula for $\mathscr{L}$ can be rewritten as follows:

$$\delta\mathscr{L} = \Delta_a \mathscr{L}' \delta y^a + \mathrm{d}_\mu \left[\Delta^\mu_a \mathscr{L}' \delta y^a + \Delta^{\mu\nu}_a \mathscr{L}' \nabla_\nu \delta y^a + \sum_{s=2}^{k-1} \Delta^{\mu\nu_1\ldots\nu_s}_a \mathscr{L}' \nabla_{\nu_1\ldots\nu_s} \delta y^a \right] \tag{15}$$

where the 'covariant' variational derivatives $\Delta_a\mathscr{L}', \Delta^\mu_a\mathscr{L}', \ldots, \Delta^{\mu_1\ldots\mu_k}_a\mathscr{L}'$ are recursively defined by

$$\Delta^{\mu_1\ldots\mu_k}_a \mathscr{L}' = D^{\mu_1\ldots\mu_k}_a \mathscr{L}' \qquad \Delta^{\mu_1\ldots\mu_{k-1}}_a \mathscr{L}' = D^{\mu_1\ldots\mu_{k-1}}_a \mathscr{L}' - \nabla_\nu \Delta^{\mu_1\ldots\mu_{k-1}\nu}_a \mathscr{L}'$$

$$\ldots \qquad \Delta^\mu_a \mathscr{L}' = D^\mu_a \mathscr{L}' - \nabla_\nu \Delta^{\mu\nu}_a \mathscr{L}' \qquad \Delta_a \mathscr{L}' = D_a \mathscr{L}' - \nabla_\mu \Delta^\mu_a \mathscr{L}'. \tag{16}$$

Here the partial 'covariant' derivatives $D_\mu\mathscr{L}', D_a\mathscr{L}', D^\mu_a\mathscr{L}', \ldots, D^{\mu_1\ldots\mu_k}_a\mathscr{L}'$ are uniquely defined by

$$\mathrm{d}\mathscr{L}' = D_\mu \mathscr{L}' \, \mathrm{d}x^\mu + D_a \mathscr{L}' \, \mathrm{d}y^a + D^\mu_a \mathscr{L}' \, \mathrm{d}\nabla_\mu y^a + \ldots + D^{\mu_1\ldots\mu_k}_a \mathscr{L}' \, \mathrm{d}\nabla_{\mu_1\ldots\mu_k} y^a \tag{17}$$

and by the requirement that $D^{\mu_1\ldots\mu_s}_a \mathscr{L}'$ be symmetric in $(\mu_1 \ldots \mu_s)$ for $2 \leqslant s \leqslant k$.

Remark. A word of caution is in order here. The Lagrangian $\mathscr{L}'$ is a scalar density of weight one, so that, as far as covariant derivatives with respect to $\Gamma^\alpha_{\beta\mu}$ are concerned, the partial 'covariant' derivatives $D^{\mu_1\ldots\mu_s}_a \mathscr{L}'$, which are tensorial objects, should be treated as if they were tensor-densities of weight one. This must be taken into account when covariant derivatives are considered, e.g.

$$\nabla_\mu D^\alpha_b \mathscr{L}' = \mathrm{d}_\mu D^\alpha_b \mathscr{L}' - D^\alpha_a \mathscr{L}' A^a_{b\mu} + \Gamma^\alpha_{\beta\mu} D^\beta_a \mathscr{L}' - \Gamma^\beta_{\beta\mu} D^\alpha_a \mathscr{L}'.$$

The variational derivatives $\Delta_a\mathscr{L}'$, $\Delta_a^\mu\mathscr{L}', \ldots, \Delta_a^{\mu_1\cdots\mu_k}\mathscr{L}'$ depend in a very peculiar way on the two connections $A^a_{b\mu}$ and $\Gamma^\alpha_{\beta\mu}$ we used to construct them. More precisely, we have:

- $\Delta_a\mathscr{L}'$ does not depend on $A^a_{b\mu}$ and $\Gamma^\alpha_{\beta\mu}$ since the identity $\Delta_a\mathscr{L}' \equiv \delta_a\mathscr{L}$ holds independently of the order k of the Lagrangian;
- the globally well defined vector density

$$\Psi^\mu(\mathscr{L}') = \Delta_a^\mu\mathscr{L}'\delta y^a + \Delta_a^{\mu\nu}\mathscr{L}'\nabla_\nu\delta y^a + \sum_{s=2}^{k-1}[\Delta_a^{\mu\nu_1\cdots\nu_s}\mathscr{L}'\nabla_{\nu_1\ldots\nu_s}\delta y^a]$$

does not depend on the connection $A^a_{b\mu}$ since the identity $\Psi^\mu(\mathscr{L}') \equiv \mathscr{F}^\mu(\mathscr{L}, \Gamma)$ holds independently of the order k of the Lagrangian (see [8] for the proof).

Accordingly, the recipe for explicitly calculating the global vector-density $\mathscr{F}^\mu(\mathscr{L}, \Gamma)$ is the following. First calculate the vector-density $\Psi^\mu(\mathscr{L}')$ as described above. Then substitute in $\Psi^\mu(\mathscr{L}')$ all the covariant derivatives $\nabla_{\nu_1\ldots\nu_s}\delta y^a$ by means of their expressions in terms of y^a, $A^a_{b\mu}$, $\Gamma^\alpha_{\beta\mu}$ and their derivatives. What we are left with at this point is exactly the explicit expression of $\mathscr{F}^\mu(\mathscr{L}, \Gamma)$.

2.2. Affine bundles

The case in which B is an affine bundle is strictly related to the discussion of section 2.1 for vector bundles. The reason is that any affine bundle $\pi : B \to M$ is modelled on a vector bundle $\boldsymbol{\pi} : \boldsymbol{B} \to M$, B and $\boldsymbol{B}$ are isomorphic as fibre bundles over M and, finally, to each global section of the affine bundle B we associate, in a canonical way, a bundle isomorphism between B and $\boldsymbol{B}$. The bundle isomorphism between B and $\boldsymbol{B}$ associated to a global section $\phi : M \to B$ is given by: $b \mapsto b - \phi(\pi(b))$. Accordingly, the natural choice for global homotopies is to choose a global section $\phi : M \to B$ and to construct the global homotopy $C : [0, 1] \times B \to B$ by setting $C(\tau, b) = \phi(\pi(b)) + \tau(b - \phi(\pi(b)))$.

The only relevant difference with the case of vector bundles, consists in the fact that we do not have any longer a preferred choice for the homotopy, because we have no preferred global section. At this point, we know that what we said for the case of vector bundles applies also to affine bundles, with the following exception: we obtain a d-invariant Lagrangian $\tilde{\mathscr{L}}(\mathscr{L}, \Gamma, \phi)$ for each global section $\phi : M \to B$.

As can be easily verified, the d-invariant Lagrangians $\tilde{\mathscr{L}}(\mathscr{L}, \Gamma, \phi')$ and $\tilde{\mathscr{L}}(\mathscr{L}, \Gamma, \phi'')$ corresponding to two different sections differ by a global scalar density having identically vanishing field equations. This means that when $H_m(M)$ does not vanish there are a Lagrangian $\mathscr{L}$ and global sections ϕ' and ϕ'', such that the difference $\tilde{\mathscr{L}}(\mathscr{L}, \Gamma, \phi') - \tilde{\mathscr{L}}(\mathscr{L}, \Gamma, \phi'')$ is not a global divergence.

2.2.1. Example: the electromagnetic field. Here we consider, as an example, the electromagnetic field in an arbitrary manifold M endowed with a Riemannian or pseudo-Riemannian metric $g_{\alpha\beta} = g_{\alpha\beta}(x)$. Since the electromagnetic potential A_μ is a connection in a principal $U(1)$-bundle over M, it is a section of an affine bundle over M which is modelled on the vector bundle $u(1) \otimes T^*M$. Accordingly, we have to apply the procedure for affine fields.

Let us then choose a background electromagnetic potential $\bar{A}_\mu = \bar{A}_\mu(x)$, and let us consider the homotopy

$$(\tau, x^\alpha, A_\mu) \mapsto (x^\alpha, \bar{A}_\mu(x) + \tau(A_\mu - \bar{A}_\mu(x))) \tag{18}$$

to calculate all the necessary integrals.

The Lagrangian $\mathscr{L}$ is the Maxwell Lagrangian

$$\mathscr{L} = -\tfrac{1}{4}\sqrt{g}\, F_{\mu\nu}F^{\mu\nu} \tag{19}$$

and the global first variation formula for $\mathscr{L}$ is

$$\delta\mathscr{L} = -\mathrm{d}_\sigma[\sqrt{g}\, F^{\mu\sigma}]\delta A_\mu + \mathrm{d}_\sigma[\sqrt{g}\, F^{\mu\sigma}\delta A_\mu] \tag{20}$$

where we set as usual $g = |\det(g_{\alpha\beta})|$, $F_{\mu\nu} = \mathrm{d}_\mu A_\nu - \mathrm{d}_\nu A_\mu$ and $F^{\mu\nu} = g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta}$. The vector-density $\tilde{\mathscr{F}}^\mu$ and the d-invariant Lagrangian $\tilde{\mathscr{L}}$ corresponding to the background field $\bar{A}_\nu$ are then given by

$$\begin{aligned}\tilde{\mathscr{F}}^\mu(\mathscr{L}, \bar{A}) &= \int_0^1 \{\sqrt{g}\,(A_\sigma - \bar{A}_\sigma)[\bar{F}^{\sigma\mu} + \tau(F_{\sigma\mu} - \bar{F}_{\sigma\mu})]\}\,\mathrm{d}\tau \\ &= \tfrac{1}{2}\sqrt{g}\,(A_\sigma - \bar{A}_\sigma)(\bar{F}^{\sigma\mu} + F^{\sigma\mu}) \end{aligned} \tag{21}$$

$$\begin{aligned}\tilde{\mathscr{L}}(\mathscr{L}, \bar{A}) &= \int_0^1 \{-(A_\sigma - \bar{A}_\sigma)\,\mathrm{d}_\mu[\sqrt{g}\,(\bar{F}^{\sigma\mu} + \tau(F_{\sigma\mu} - \bar{F}_{\sigma\mu}))]\}\,\mathrm{d}\tau \\ &= -\tfrac{1}{2}(A_\sigma - \bar{A}_\sigma)\,\mathrm{d}_\mu[\sqrt{g}\,(\bar{F}^{\sigma\mu} + F^{\sigma\mu})]. \end{aligned} \tag{22}$$

It can be now easily checked that two background connections $\bar{A}'_\sigma$ and $\bar{A}''_\sigma$ give rise to two Lagrangians (22) which differ by a quantity which is a global divergence if and only if

$$\tfrac{1}{2}\partial_\mu[\sqrt{g}\,(\bar{F}''^{\sigma\mu} + \bar{F}'^{\sigma\mu})](\bar{A}''_\sigma - \bar{A}'_\sigma)$$

is a global divergence. This may be true in some special cases, but in general it is false.

Conclusions

We have seen that our identity admits a global generalization in the case of vector bundles and affine bundles, which cover most cases of physical interest. The methods described in this paper can be easily applied to fibre bundles $\pi : B \to M$ which admit a global section $\phi : M \to B$ such that the map $\phi \circ \pi : B \to B$ is homotopic to the identity. In this case we lose uniqueness of the result, because we have no preferred homotopy. Whether we lose the possibility of constructing a global vector-density $\mathscr{F}^\mu$ depends, as in the previous cases, on the cohomology group $H_m(M)$ and also on the Lagrangian $\mathscr{L}$.

In the case of fibre bundles with non-trivial topology, uniqueness and globality of the result are lost usually.

Acknowledgments

The work is partially supported by MURST, through 40% research project 'Geometria e Fisica' and Gruppo Nazionale per la Fisica Matematica, Consiglio Nazionale delle Ricerche, Italy. One of us (VT) acknowledges a visiting professorship grant by GNFM-CNR.

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